Exercise 13

Solve the initial-value problem.

$$y'' - 5y' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$$

Solution

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Substitute these formulas into the ODE.

$$r^2 e^{rx} - 5(re^{rx}) + 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 5r + 4 = 0$$

Solve for r.

$$(r-4)(r-1) = 0$$

$$r = \{1, 4\}$$

Two solutions to the ODE are e^x and e^{4x}. According to the principle of superposition, the general solution is a linear combination of these two.

$$y(x) = C_1 e^x + C_2 e^{4x}$$

Differentiate it with respect to x.

$$y'(x) = C_1 e^x + 4C_2 e^{4x}$$

Apply the initial conditions to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 0$$

$$y'(0) = C_1 + 4C_2 = 1$$

Solve the system.

$$C_1 = -\frac{1}{3} \qquad C_2 = \frac{1}{3}$$

Therefore,

$$y(x) = -\frac{1}{3}e^x + \frac{1}{3}e^{4x}.$$

Below is a plot of the solution versus x.